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SOLVING A NONLINEAR INVERSE PROBLEM OF IDENTIFYING AN UNKNOWN SOURCE TERM IN A REACTION-DIFFUSION EQUATION BY ADOMIAN DECOMPOSITION METHOD

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Abstract. We consider the inverse problem of finding the nonlinear source for nonlinear Reaction-Diffusion equation whenever the initial and boundary condition are given. We investigate the numerical solution of this problem by using Adomian Decomposition Method (ADM). The approach of the proposed method is to approximate unknown coefficients by a nonlinear function whose coefficients are determined from the solution of minimization problem based on the overspecified data. Further, the Tikhonov regularization method is applied to deal with noisy input data and obtain a stable approximate solution. This method is tested for two examples. The results obtained show that the method is efficient and accurate. This study showed also, the speed of the convergent of ADM.

Keywords: Inverse problem; Adomian Decomposition Method (ADM); Convergence; Overspecified data; Least Square; Tikhonov Regularization Method.

AMS Subject Classification: 65M32, 35K05.

1. Introduction

The Adomian Decomposition method was called after its creator: Gorge Adomian [1], [2], [3], [13]. The method is useful for solving a variety of problems. A review of the application of the Adomian decomposition method for solving differential and integral equations was discussed in [2,3]. It was also used for solving the linear and nonlinear heat transfer equation in [25], whereas its use for solving the wave equation was tested in [14]. In [15] the ADM was utilized for solving the inverse problems of differential equations. The method may also be employed in mathematical models describing different technical problems as discussed in [8]. The proof of the convergence for the ADM was discussed by charrault [9].

Recovery of missing parameters in partial differential equations from overspecified data plays an important role in inverse problems arising in engineering and physics. These problems are widely encountered in the modelling of interesting phenomena, e.g. heat conduction and hydrology. Another challenge of mathematical modelling is to determine

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what additional information is necessary and/or sufficient to ensure the (unique) solvability of an unknown physical parameter of a given process. a variety of techniques for solving inverse problems have been proposed [22]- [24] Inverse source problem arises from many physical and engineering problems. Inverse source problem is known to be ill-posed in the sense that a small change in the given data may result in a dramatic change in the solution. For inverse source identification problem, there have been a lot of research results, e.g. the existence and uniqueness of the solution, [7] the conditional stability and data compatibility of the solution, [18] numerical algorithms for the identification problem. [11] Due to the ill-posedness of the problem, numerical computation is difficult. In order to obtain a stable numerical solution for this kind of ill-posed problem, some special methods are required such as Tikhonov regularization [26], iterative regularization [4], mollification [17], BFM (Base Function Method) [23], SFDM (Semi Finite Difference Method) [16] and the FSM (Function Specification Method) [5].

In this present paper we solve the problem of structural identification of an unknown source term in a class of Reaction-Diffusion equation. This equation is used to model a wide range of phenomena in physics, engineering, chemistry and biology [19]. This problem is described by the following inverse problem: Find $u = u(x,t)$ and $F = F(u)$ which satisfy

$$u_t(x,t) = (A(u)u_x)_x + F(u),\ (x,t) \in Q_T = (0,1) \times (0,T), \tag{1.1}$$

$$u(x,0) = f(x),\ x \in (0,1), \tag{1.2}$$

$$u(0,t) = p(t),\ t \in (0,T), \tag{1.3}$$

$$u(1,t) = q(t),\ t \in (0,T). \tag{1.4}$$

Subject to the overspecification

$$u(\alpha,\beta) = k,\ \alpha \in (0,1),\ \beta \in (0,T), \tag{1.5}$$

Where $f(x)$, $p(t)$, $q(t)$ and k are known function and $A(u)$, is arbitrary smooth function. In this context of heat conduction and diffusion when u represent temperature and concentration, the unknown function $F(u)$ is interpreted as a heat and material source, respectively, while in a chemical or biochemical application F may be interpreted as a reaction term. Although the results in this paper apply to each of these interpretations, the unknown function $F(u)$ will be referred to here as a source term. The approach to solve this problem referred to in the literature as a method of output least squares in to assume that the unknown function is a specific functional form depending on some parameters and then seek to determine optimal parameter values so as to minimize an error functional based on the overspecified data.

The organization of this paper is as the following: we give a brief analysis of the method in section 2. Moreover, we compare the numerical results with the exact solutions and those were obtained by the ADM in section 3 and section 4 consists of a brief conclusions.

2. Analysis of the method

In the section we are considering the following inverse parabolic problems

$$u_t = (A(u)u_x)_x + F(u), \qquad (x,t) \in Q_T = (0,1) \times (0,T), \tag{2.1a}$$

$$u(x,0) = f(x), \qquad x \in (0,1), \tag{2.1b}$$

$$u(0,t) = p(t), \qquad t \in (0,T), \tag{2.1c}$$

$$u(1,t) = q(t), \qquad t \in (0,T), \tag{2.1d}$$

Where $f(x)$, $p(t)$, $q(t)$ are given functions, the indices t and x denote differentiating with respect to these variables. $A(u)$ is the arbitrary function, it is concentration dependent conductivity. Such problem arise in plasma and solid state physics and polymer sciences. Let us consider the linear and nonlinear operators L, N respectively, defined by

$$L(u) = \frac{\partial u}{\partial t},$$

$$N(u) = \frac{\partial}{\partial x}(A(u)u_x) + F(u) = \frac{\partial((A(u))}{\partial x}u_x + A(u)\frac{\partial^2 u}{\partial x^2} + F(u),$$

Then the equation (2.1a) can be written

$$L(u) = N(u),$$

The operator L is invertible and

$$L^{-1}(.) = \int_0^t .\ dt,$$

if we operate L^{-1} in the right hand side of (2.1a) and use the initial condition $u(x,0) = f(x)$, we have

$$u(x,t) = f(x) + L^{-1}(N(u(x,t)),$$

Application to (2.1a) gives

$$u(x,t) = f(x) + \int_0^t ((A(u)u_x)_x + F(u))dt, \tag{2.2}$$

Remark 2.1. In this work the polynomial form proposed for the unknown $F(u)$ before performing the calculation, therefore $F(u)$ is approximated as

$$F(u) = a_0 + a_1 u + a_2 u^2 + ... + a_m u^m,$$

where $\{a_0, a_1, ..., a_m\}$ are constants which remain to be determined by least squares method.

Adomians method consists in calculating the solution u of (2.2) in a series form

$$u = \sum_{n=0}^{\infty} u_n, \tag{2.3}$$

And the nonlinear term becomes

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{2.4}$$

Where A_n called Adomian polynomials [9], has been introduced by the adomian himself by the formula

$$A_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{n} \lambda^i u_i)]_{\lambda=0},\ n = 0, 1, 2, ...,$$

Putting equation (2.3) and (2.4) into (2.2) gives

$$\sum_{n=0}^{\infty} u_n = f(x) + \int_0^t (A(u)u_x)_x + F(u)dt,$$

Each term of series $\sum_{n=0}^{\infty}$ can be identified by the formula

$$\begin{cases} u_0(x,t) = f(x), \\ u_{n+1}(x,t) = L^{-1}(A_n),\ n \geqslant 0, \end{cases}$$

And the A_n are determined by using the relationship

$$A_0 = a_0 + a_1u_0 + a_2u_0^2 + ... + a_mu_0^m + A'(u_0)^2(u_{0x})^2 + A(u_0)u_{0xx},$$

$$A_1 = 2A'(u_0)u_{0x}u_{1x} + u_1(a_1 + 2a_2u_0 + ... + ma_mu_0^{m-1} + A''(u_0)(u_{0x})^2 + A'(u_0)u_{0xx}) + A(u_0)u_{1xx},$$

$$A_2 = \frac{1}{2}(2u_2(a_1 + 2a_2u_0 + ... + ma_mu_0^{m-1} + A''(u_0)(u_{0x})^2 + A'(u_0)u_{0xx})$$
$$+u_1^2(2a_2 + ... + m(m-1)a_mu_0^{m-2} + A^{(3)}(u_0)(u_{0x})^2 + A''(u_0)u_{0xx})$$
$$+2u_1(2A''(u_0)u_{0x}u_{1x} + A'(u_0)u_{1xx}) + 2(A'(u_0)((u_{1x})^2 + 2u_{0x}u_{2x}) + A(u_0)u_{2xx}))$$
$$\vdots$$

In practice, all the terms of the series $\sum_{n=0}^{\infty} u_n$ cannot be determined, and we use an approximation of the solution u from the new series

$$\phi_n = \sum_{i=0}^{n-1} u_i, \ with \ \lim_{n\to\infty} \phi_n = u.$$

2.1. **least-squares minimization technique.** The estimated confiscations a_Υ, $\Upsilon = 0, 1, ..., m$ can be determined by using least squares method when the sum of the squares of the deviation between the calculated $u(\alpha, \beta), u(0, \beta), u(1, \beta)$ by ADM and the measured $k, p(\beta), q(\beta)$ of (1.5) and (1.3) and (1.4) at $\alpha \in (0, 1)$, $\beta \in (0, T)$ is less than a small number. The error in the estimates $E(a_0, ..., a_m)$ can be expressed as

$$E(a_0, ..., a_m) = (u(\alpha, \beta) - k)^2 + (u(0, \beta) - p(\beta))^2 + (u(1, \beta) - q(\beta))^2,$$

which is to be minimized. To obtain the minimum value of $E(a_0, ..., a_m)$ with respect to $a_0, ..., a_m$, differentiation of $E(a_0, ..., a_m)$, with respect to $a_0, ..., a_m$, will be performed. Thus the system corresponding to the value of $a_0, ..., a_m$ can be solved, but when $u(\alpha, \beta)$ affected by measurement errors, the estimate of $a_0, ..., a_m$ will be unstable so that we used the Tikhonov regularization method for controlling this measurement errors ([10], [12]).

3. Applications and Results

In this section, we are going to study numerically the inverse problems (1.1)-(1.4) with the unknown source. The main aim here is to show the applicability of the present method, described in Section 2, for solving the inverse problems (1.1)-(1.4). The inverse problems are ill-posed and therefore it is necessary to investigate the stability of the present method by giving a test problem.

Remark 3.1. In an inverse problem there are two sources of error in the estimation. The first source is the unavoidable bias deviation (or deterministic error). The second source of error is the variance due to the amplification of measurement errors (stochastic error). The global effect of deterministic and stochastic errors is considered in the mean squared error or total error, [6].

Therefore, we compute total error S by using following formulae

$$S = \left[\frac{1}{N-1}\sum_{i=1}^{N}(u_{Exact}(x_j, t_i) - u_{ADM}(x_j, t_i))^2\right]^{\frac{1}{2}}, \ \forall j \in \{1, 2, ..., \}, \tag{3.1}$$

where N is the total number of estimated values.

Example 1. *In this example we determine $u = u(x,t)$ and $F(u)$ satisfying*

$$u_t = (au\ exp(au)u_x)_x + F(u),\ 0 \le x \le 1,\ 0 \le t \le T. \tag{3.2}$$

with given data

$$u(x,0) = \frac{1}{\delta} log(C_1 x + C_2),\ 0 \le x \le 1,\ C_1, C_2 \in \mathbb{R},$$

$$u(0,t) = \frac{1}{\delta} log((\frac{aC_1^2}{\delta} + b\delta)t + C_2),\ 0 \le t \le T,$$

$$u(1,t) = \frac{1}{\delta} log(C_1 + (\frac{aC_1^2}{\delta} + b\delta)t + C_2),\ 0 \le t \le T.$$

The exact solution of this problem is

$$u(x,t) = \frac{1}{\delta} log(C_1 x + (\frac{aC_1^2}{\delta} + b\delta)t + C_2), 0 \le x \le 1,\ 0 \le t \le T,\ C_1, C_2 \in \mathbb{R},$$

$$F(u(x,t)) = b\ exp(-\delta\ u).$$

By using ADM and Least square we estimate a_0, a_1, a_2 for $F(u) = a_0 + a_1 u + a_2 u^2$, we have

$$a_0 = 3.6008997282536794343512732204667,$$

$$a_1 = 0.020729696230948750214061070156708,$$

$$a_2 = 0.000032732678806070674669042788405756,$$

The results obtained for $u(0.2,t)$ and $F(u(0.8,t))$ with $T = 1$ with noisy data are presented in Table 1, 2 and Figure 1, 2.

t	$Exact\ u(0.2,t)$	$Numerical\ u(0.2,t)$
0.100000	−442.118335	−442.118086
0.200000	−441.952084	−441.951523
0.300000	−441.786109	−441.785178
0.400000	−441.620409	−441.619051
0.500000	−441.454983	−441.453144
0.600000	−441.289830	−441.287461
0.700000	−441.124949	−441.122003
0.800000	−440.960340	−440.956773
0.900000	−440.796002	−440.791772
1.000000	−440.631933	−440.627003
S		$2.545533187579e-003$
Execution Time (second)		15.568977

Table 1. approximate solution and exact solution when $C_1 = C_2 = \delta = a = b = 0.01$ and $\alpha = 0.5$, $\beta = 0.5$.

t	$Exact\ F(u(0.8,t))$	$Numerical\ F(u(0.8,t))$
0.100000	0.554939	0.555216
0.200000	0.554324	0.554599
0.300000	0.553710	0.553984
0.400000	0.553097	0.553371
0.500000	0.552486	0.552759
0.600000	0.551876	0.552149
0.700000	0.551268	0.551540
0.800000	0.550661	0.550933
0.900000	0.550055	0.550328
1.000000	0.549451	0.549724
S		$2.765857230446590e-004$
Execution Time (second)		15.568977

Table 2. approximate solution and exact solution when $C_1 = C_2 = \delta = a = b = 0.01$ and $\alpha = 0.5$, $\beta = 0.5$.

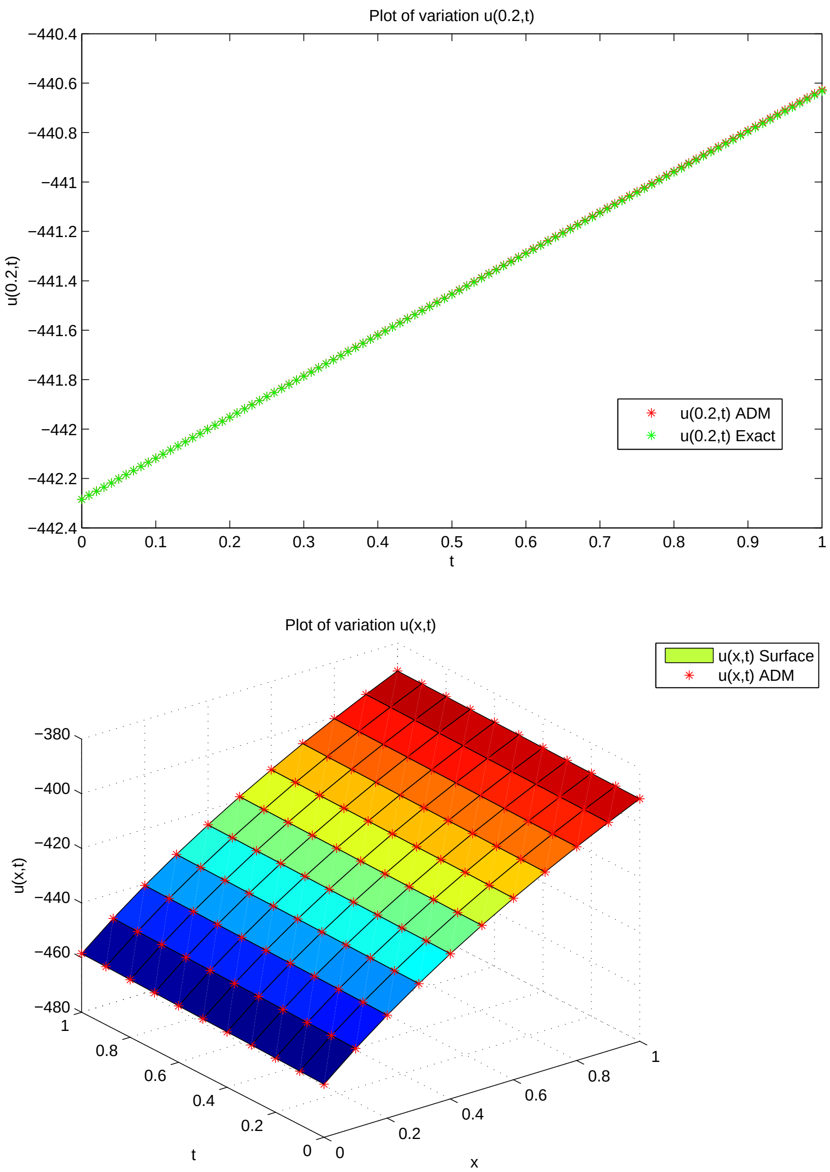


Figure 1. The comparison between the exact and numerical results for $u(x,t)$ of the problem (3.3) with the noisy data by using ADM method when $C_1 = C_2 = \delta = a = b = 0.01$ and $\alpha = 0.5$, $\beta = 0.5$.

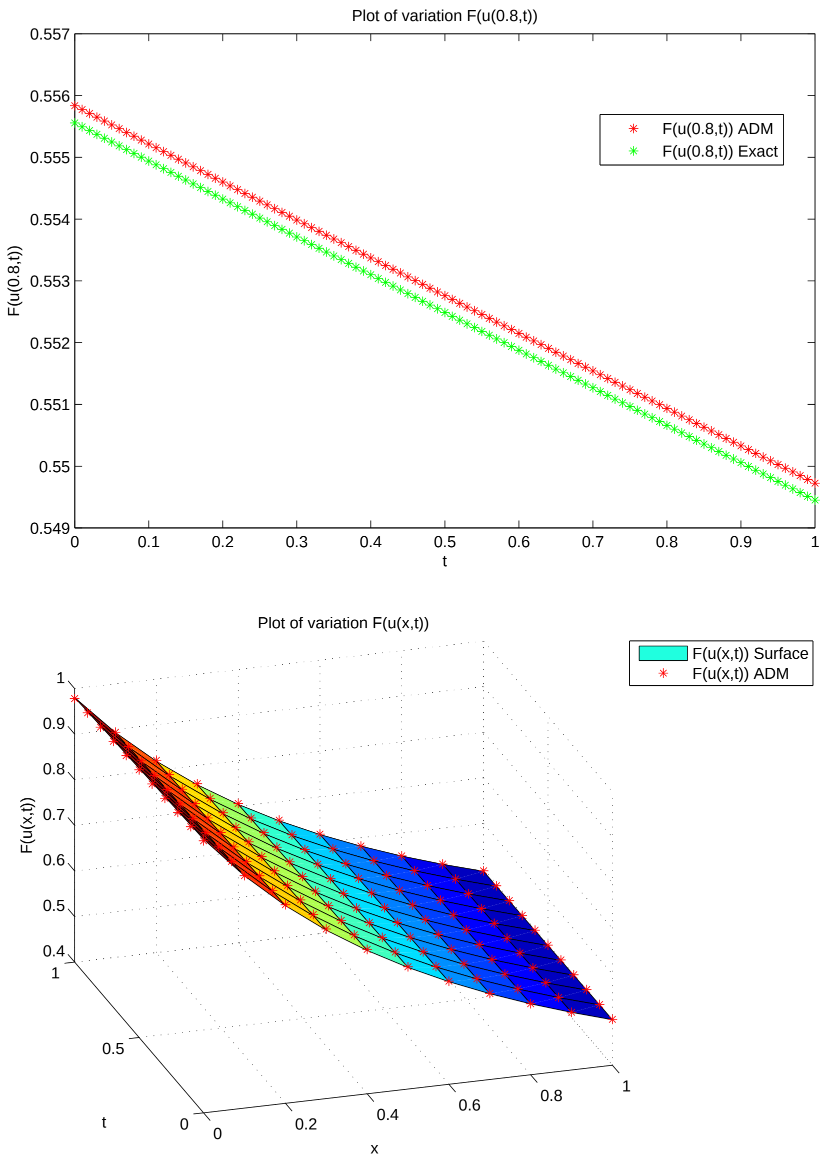


Figure 2. The comparison between the exact and numerical results for $F(u(x,t))$ of the problem (3.3) with the noisy data by using ADM method when $C_1 = C_2 = \delta = a = b = 0.01$ and $\alpha = 0.5$, $\beta = 0.5$.

Example 2. *In this example we determine* $u = u(x,t)$ *and* $F(u)$ *satisfying*

$$u_t = ((1-u)u_x)_x + F(u), \ \ 0 \le x \le 1, \ \ 0 \le t \le T. \tag{3.3}$$

with given data

$$u(x,0) = \frac{1}{3}(2 + sinx), \ \ 0 \le x \le 1,$$

$$u(0,t) = \frac{1}{3}[\frac{exp(-t)[3exp(2t)+1]}{exp(t)+exp(-t)}], \ \ 0 \le t \le T,$$

$$u(1,t) = \frac{1}{3}[\frac{exp(-t)[3exp(2t)+1+2sin1]}{exp(t)+exp(-t)}], \ \ 0 \le t \le T.$$

The exact solution of this problem is

$$u(x,t)=\frac{1}{3}[\frac{exp(-t)[3exp(2t)+1+2sinx]}{exp(t)+exp(-t)}],\ 0\le x\le 1,\ 0\le t\le T,$$

$$F(u(x,t))=2u-2u^2.$$

By using ADM and Least square we estimate a_0,a_1,a_2 *for* $F(u)=a_0+a_1u+a_2u^2$*, we have*

$$a_0=-0.0294700700446760173215552012002 46,$$

$$a_1=2.0756770803450796603765945006097,$$

$$a_2=-2.0470530306521417537840365521654,$$

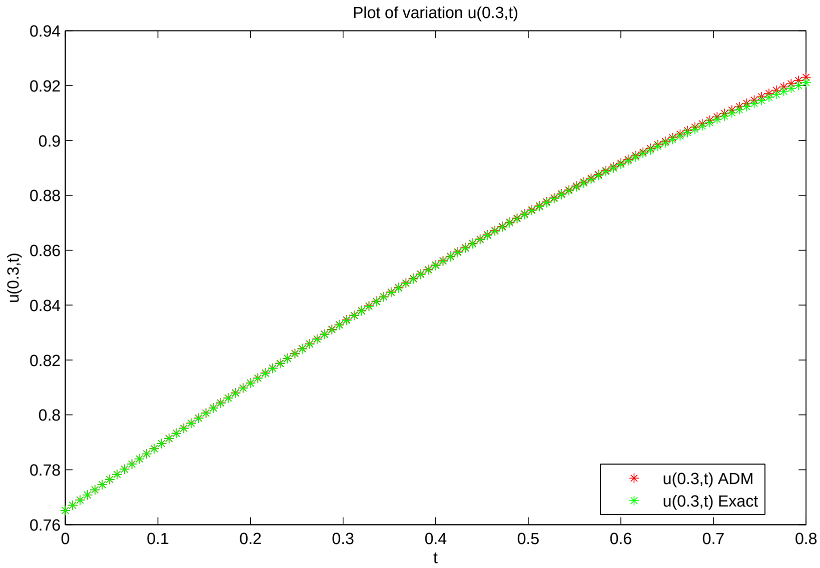
The results obtained for $u(0.3,t)$ *and* $F(u(0.7,t))$ *with* $T=0.8$ *with noisy data are presented in Table 3, 4 and Figure 3, 4.*

t	$Exact\ u(0.3,t)$	$Numerical\ u(0.3,t)$
0.080000	0.783920	0.783981
0.160000	0.802428	0.802536
0.240000	0.820474	0.820617
0.320000	0.837854	0.838024
0.400000	0.854396	0.854596
0.480000	0.869963	0.870212
0.560000	0.884460	0.884812
0.640000	0.897827	0.898406
0.720000	0.910040	0.911081
0.800000	0.921107	0.923023
S		$5.834632700437494e-004$
Execution Time (second)		18.366680

Table 3. approximate solution and exact solution when $\alpha=0.5$, $\beta=0.1$.

t	$Exact\ F(u(0.7,t))$	$Numerical\ F(u(0.7,t))$
0.080000	0.194436	0.194963
0.160000	0.179647	0.180028
0.240000	0.164891	0.165134
0.320000	0.150365	0.150480
0.400000	0.136254	0.136244
0.480000	0.122719	0.122575
0.560000	0.109892	0.109575
0.640000	0.097876	0.097285
0.720000	0.086736	0.085674
0.800000	0.076512	0.074628
S		$6.321097687149357e-004$
Execution Time (second)		18.366680

Table 4. approximate solution and exact solution when $\alpha=0.5$, $\beta=0.1$.



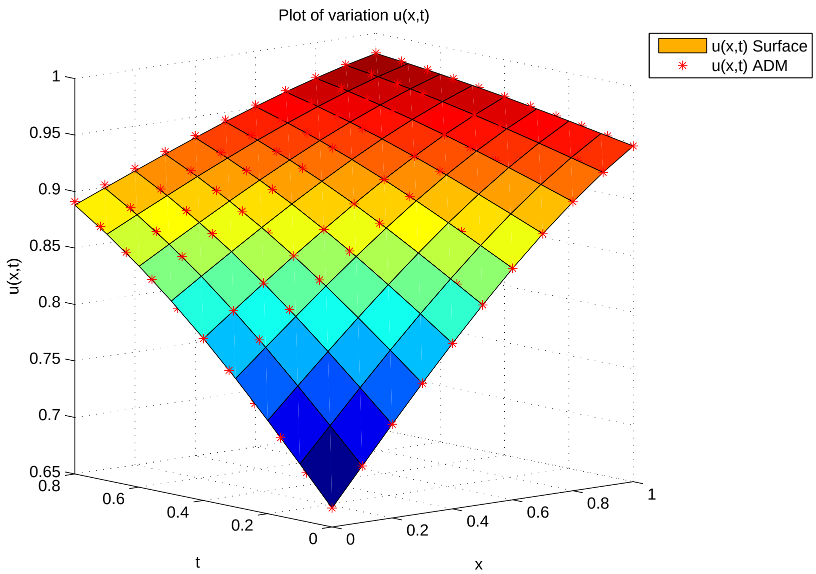


Figure 3. The comparison between the exact and numerical results for $u(x,t)$ of the problem (2) with the noisy data by using ADM method when $\alpha = 0.5$, $\beta = 0.1$.

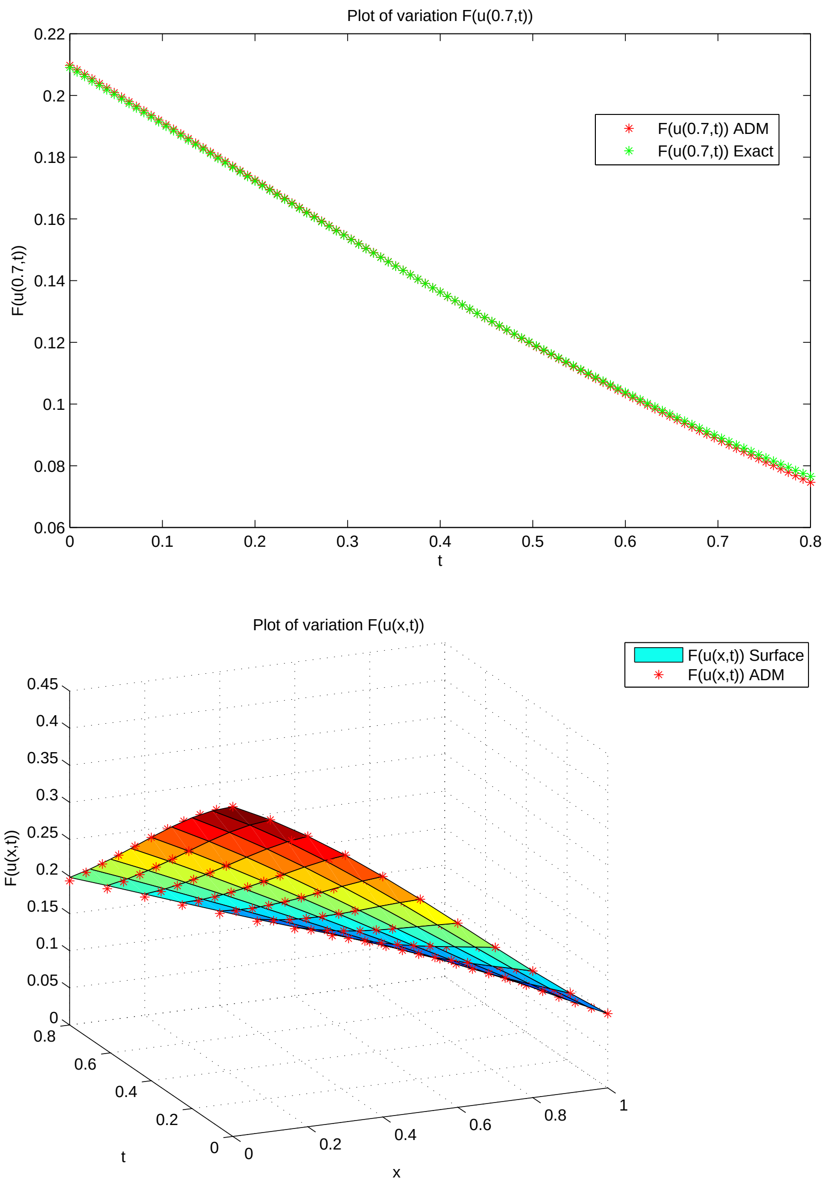


Figure 4. The comparison between the exact and numerical results for $F(u(x,t))$ of the problem (2) with the noisy data by using ADM method when $\alpha = 0.5$, $\beta = 0.1$.

4. Conclusion

An advantage of this method is that, it can provide analytical or an approximated solution to a rather wide class of nonlinear (and stochastic) problems without linearization, perturbation, closure approximation, or discretization method. Unlike the common methods which are only applicable to systems with weak nonlinearity and small perturbation and many changes the physics of the problem due to simplification, ADM gives the approximated solution of the problem without any simplification. Thus, its results are more realistic. In this study, we showed that ADM applies to inverse problems successfully and numerical results show that an excellent estimation can be obtained within a couple of minutes CPU time at pentium(R) 4 CPU 2.10 GHz.

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